

Notes on the existence of solutions in the pairwise comparisons method using the Heuristic Rating Estimation approach

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Abstract Pairwise comparisons are a well-known method for modelling of the subjective preferences of a decision maker. A popular implementation of the method is based on solving an eigenvalue problem for M - the matrix of pairwise comparisons. This does not take into account the actual values of preference. The Heuristic Rating Estimation (*HRE*) approach is a modification of this method in which allows modelling of the reference values. To determine the relative order of preferences is to solve a certain linear equation system defined by the matrix A and the constant term vector b (both derived from M).

The article explores the properties of these equation systems. In particular, it is proven that for some small data inconsistency the A matrix is an M -matrix, hence the equation proposed by the *HRE* approach has a unique strictly positive solution.

1 Introduction

The first written evidence of the use of pairwise comparisons (*PC*) dates back to the thirteenth century [4]. After a period of growth in the first half of the twentieth century, the pairwise comparisons method solidified in the form of the Analytic Hierarchy Process (*AHP*) proposed by Saaty [19]. Starting as a voting method, *PC* has become a way of deciding on the relative importance (relative utility) of concepts (alternatives), used in decision theory [19], economics [16], psychometrics and psychophysics [20] and others. The utility of the method has been confirmed many times by various researchers [7]. The theory of paired comparison is growing all the time. Examples of such exploration are the *Rough Set* approach [6], fuzzy *PC* relation handling [15], incomplete *PC* relation [1, 5, 11], data inconsistency reduction [12] and non-numerical rankings [9]. A recent contribution to the pairwise comparisons method includes the *Heuristic Rating Estimation (HRE)* approach [13, 14] that allows the user to explicitly define a reference set of concepts, for which the utilities (the ranking values) are known a priori. The base heuristics used in *HRE* proposes to determine the relative values of a single non-reference concept as a weighted average of all the other concepts. Such a proposition leads to formulation a linear equation system defined by the matrix A and the strictly positive vector of constant terms b . As will be shown later, in the most interesting cases the matrix A is an M -matrix as defined in [17]. The sufficient condition for A to be an M -matrix is formulated using the notion of inconsistency referring to the quantitative relationship between entries of the pairwise comparisons matrix W . In particular it is shown that the fully consistent *PC* matrix W implies that A is an M -matrix.

2 Preliminaries

2.1 Pairwise comparisons method

The input to the *PC* method is $M = (m_{ij}) \wedge m_{ij} \in \mathbb{R}_+ \wedge i, j \in \{1, \dots, n\}$ a *PC* matrix that expresses a quantitative relation R over the finite set of concepts $C \stackrel{\text{df}}{=} \{c_i \in \mathcal{C} \wedge i \in \{1, \dots, n\}\}$ where \mathcal{C} is a non empty universe of concepts and $R(c_i, c_j) = m_{ij}$, $R(c_j, c_i) = m_{ji}$. The values m_{ij} and m_{ji} represent subjective expert judgment as to the relative importance, utility or quality indicators of the concepts c_i and c_j . Thus, according to the best knowledge of experts it should hold that $c_i = m_{ij}c_j$.

Definition 1. A matrix M is said to be reciprocal if for every $i, j \in \{1, \dots, n\}$ holds $m_{ij} = \frac{1}{m_{ji}}$, and M is said to be consistent if for every $i, j, k \in \{1, \dots, n\}$ holds $m_{ij} \cdot m_{jk} \cdot m_{ki} = 1$.

Since the data in the *PC* matrix represents the subjective opinions of experts, they might be inconsistent. Hence, there may exist a triad m_{ij}, m_{jk}, m_{ki} of entries in M for which $m_{ik} \cdot m_{kj} \neq m_{ij}$. This leads to a situation in which the relative importance of c_i with respect to c_j is either $m_{ik} \cdot m_{kj}$ or m_{ij} . This observation gives rise to both a priority deriving method that transforms even an inconsistent matrix of pairwise comparisons into a consistent priority vector, and an inconsistency index describing to what extent the matrix M is inconsistent. There are a number of priority deriving methods and inconsistency indexes [2, 8, 3]. For the purpose of the article the *Koczkodaj's inconsistency index* is adopted [10].

Let us denote:

$$\kappa_{i,j,k} \stackrel{df}{=} \min \left\{ \left| 1 - \frac{m_{ij}}{m_{ik}m_{kj}} \right|, \left| 1 - \frac{m_{ik}m_{kj}}{m_{ij}} \right| \right\} \quad (1)$$

Definition 2. *Koczkodaj's inconsistency index \mathcal{K} of $n \times n$ and $(n > 2)$ reciprocal matrix M is equal to*

$$\mathcal{K}(M) = \max_{i,j,k \in \{1, \dots, n\}} \{ \kappa_{i,j,k} \} \quad (2)$$

where $i, j, k = 1, \dots, n$ and $i \neq j \wedge j \neq k \wedge i \neq k$.

The result of the pairwise comparisons method is a ranking - a function that assigns values to concepts. Formally, it can be defined as follows:

Definition 3. *The ranking function for C (the ranking of C) is a function $\mu : C \rightarrow \mathbb{R}_+$ that assigns to every concept from $C \subset \mathcal{C}$ a positive value from \mathbb{R}_+ .*

Thus, $\mu(c)$ represents the ranking value for $c \in C$. The μ function is usually defined as a vector of weights

$$\mu \stackrel{df}{=} [\mu(c_1), \dots, \mu(c_n)]^T \quad (3)$$

. According to the most popular eigenvalue based approach, proposed by *Saaty* [19], the final ranking μ_{ev} is determined as the principal eigenvector of the *PC* matrix M , rescaled so that the sum of all its entries is 1, i.e.

$$\mu_{ev} = \left[\frac{\mu_{max}(c_1)}{s_{ev}}, \dots, \frac{\mu_{max}(c_n)}{s_{ev}} \right]^T \text{ and } s_{ev} = \sum_{i=1}^n \mu_{max}(c_i) \quad (4)$$

where μ_{ev} - the ranking function, μ_{max} - the principal eigenvector of M . A more complete overview including other methods can be found in [2, 8].

2.2 Heuristic Rating Estimation approach

In the classical pairwise comparisons method the ranking function μ for all the concepts $c \in C$ is initially unknown. Hence every $\mu(c)$ need to be determined by the priority deriving procedure. In real life, however, may happen that for some concepts the priority values are known. Sometimes decision makers have extra knowledge about the group of elements $C_K \subseteq C$ that allows them to determine $\mu(c)$ for C_K in advance.

For example, let c_1, c_2 and c_3 be a goods that the company X intends to place on the market, whilst c_4 and c_5 have been available for some time in stores. In order to choose the most profitable and promising product out of c_1, \dots, c_3 the company X want to calculate the function μ for c_1, c_2 and c_3 . Due to some similarities between c_1, \dots, c_3 and the pair c_4, c_5 the company X want to include them in the ranking treating as a reference. Of course it makes no sense to ask experts about how profitable c_4 and c_5 are. The values $\mu(c_4)$ and $\mu(c_5)$ can be easily determined based on sales reports.

The situation as outlined in this simple example leads to the *Heuristic Rating Estimation (HRE)* model proposed in [13, 14]. The main heuristics of the *HRE* model assume that the set of concepts C is composed of the unknown concepts $C_U = \{c_1, \dots, c_k\}$ and known (reference) concepts $C_K = \{c_{k+1}, \dots, c_n\}$. Of course, only the values μ_j for $c \in C_U$ need to be estimated, whilst the values $\mu(c_i)$ for $c_i \in C_K$ are considered to be known. The adopted heuristics assumes that for every unknown $c_j \in C_U$ the value $\mu(c_j)$ should be estimated as the arithmetic mean of all the other values $\mu(c_i)$ multiplied by factor m_{ji} :

$$\mu(c_j) = \frac{1}{n-1} \sum_{i=1, i \neq j}^n m_{ji} \mu(c_i) \quad (5)$$

Thus, the value $\mu(c_i)$ for each unknown concept $c_j \in C_U$ is calculated according to the following formulas:

$$\begin{aligned}\mu(c_1) &= \frac{1}{n-1}(m_{2,1}\mu(c_2) + \dots + m_{n,1}\mu(c_n)) \\ \mu(c_2) &= \frac{1}{n-1}(m_{1,2}\mu(c_1) + m_{3,2}\mu(c_3) + \dots + m_{n,2}\mu(c_n)) \\ &\dots\dots\dots \\ \mu(c_k) &= \frac{1}{n-1}(m_{1,k}\mu(c_1) + \dots + m_{k-1,k}\mu(c_{k-1}) + \\ &\quad + m_{k+1,k}\mu(c_{k+1}) + \dots + m_{n,k}\mu(c_n))\end{aligned}\tag{6}$$

Since the values $\mu(c_{k+1}), \dots, \mu(c_n)$ are known and constant (c_{k+1}, \dots, c_n are the reference concepts), they can be grouped together. Let us denote:

$$b_j = \frac{1}{n-1}m_{k+1,j}\mu(c_{k+1}) + \dots + \frac{1}{n-1}m_{n,j}\mu(c_n)\tag{7}$$

Thus (6) could be written as the linear equation system $A\mu = b$ where the matrix A is:

$$A = \begin{bmatrix} 1 & \dots & -\frac{1}{n-1}m_{1,k} \\ -\frac{1}{n-1}m_{2,1} & \dots & -\frac{1}{n-1}m_{2,k} \\ \vdots & \ddots & \vdots \\ -\frac{1}{n-1}m_{k,1} & \dots & 1 \end{bmatrix},\tag{8}$$

and the vectors b and μ are:

$$b = \begin{bmatrix} \frac{1}{n-1} \sum_{i=k+1}^n m_{1,i}\mu(c_i) \\ \frac{1}{n-1} \sum_{i=k+1}^n m_{2,i}\mu(c_i) \\ \vdots \\ \frac{1}{n-1} \sum_{i=k+1}^n m_{k,i}\mu(c_i) \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu(c_1) \\ \mu(c_2) \\ \vdots \\ \mu(c_k) \end{bmatrix}\tag{9}$$

It is worth noting that $b > 0$, since every b_i for $i = 1, \dots, k$ is a sum of strictly positive components. According to (Def. 3) the ranking results must be strictly positive, hence only strictly positive vectors μ are considered to be feasible.

2.3 M-matrices

The answer to the question concerning the existence of solution of the linear equation system $A\mu = b$ requires knowledge of certain properties of the *M-matrix* [18]. For this purpose, let us denote $\mathcal{M}_{\mathbb{R}}(n)$ - the set of $n \times n$ matrices over \mathbb{R} , $\mathcal{M}_{\mathbb{Z}}(n)$ - the set of all $A = [a_{ij}] \in \mathcal{M}_{\mathbb{R}}(n)$ with $a_{ij} \leq 0$ if $i \neq j$ and $i, j \in \{1, \dots, n\}$. Moreover, for every matrix $A \in \mathcal{M}_{\mathbb{R}}(n)$ and vector $b \in \mathbb{R}^n$ the notation $A \geq 0$ and $b \geq 0$ will mean that each entry of A and b is non-negative and neither A nor b equals 0. The spectral radius of A is defined as $\rho(A) = \max\{|\lambda| : \det(\lambda I - A) = 0\}$.

Definition 4. An $n \times n$ matrix that can be expressed in the form $A = sI - B$ where $B = [b_{ij}]$ with $b_{ij} \geq 0$ for $i, j \in \{1, \dots, n\}$, and $s \geq \rho(B)$, the maximum of the moduli of the eigenvalues of B , is called *M-matrix*.

In practice, solving many problems in the biological and social sciences can be reduced to problems, involving *M-matrices* [17]. For this reason, *M-matrices* have been of interest to researchers for a long time and many of their properties have already been proven. Following [17] some of them are recalled below in the form of the Theorem 1.

Theorem 1. For every $A \in \mathcal{M}_{\mathbb{Z}}(n)$ each of the following conditions is equivalent to the statement: A is a nonsingular *M-matrix*.

1. A is inverse positive. That is, A^{-1} exists and $A^{-1} \geq 0$
2. A is semi-positive. That is, there exists $x > 0$ with $Ax > 0$
3. There exists a positive diagonal matrix D such that AD has all positive row sums.

In the context of equation $A\mu = b$ it is worth noting that if A is nonsingular then A^{-1} is also nonsingular, and thus the vector μ could be determined as $A^{-1}b$. Moreover for $b > 0$ (every entry of vector b is a sum of strictly positive values) and A - *M-matrix*, due to the theorem above $A^{-1} \geq 0$, the vector μ also must be strictly positive i.e. $\mu = A^{-1}b > 0$.

3 Inconsistency based condition for the existence of a solution

The entries of $M = [m_{ij}]$ represent comparative opinions of experts, they are thus inherently strictly positive, that is $M > 0$. For the same reason the matrix A (8), formed on the basis of M , has positive entries only on the diagonal, i.e. $A \in \mathcal{M}_{\mathbb{Z}}(n)$. Therefore proving that A satisfies any of the conditions of the Theorem 1, implies that A is an M -matrix. Hence, due to the remarks below the Theorem 1, and the fact that in the HRE approach $b > 0$, the equation $A\mu = b$ has only one strictly positive solution μ .

The sufficient condition for A to be an M -matrix is formulated with the help of the inconsistency index $\mathcal{K}(M)$ (Def. 2). Using an inconsistency index simplifies the evaluation of $A\mu = b$ and enables linking the reliability of expert assessments (the paired ranking for which the inconsistency index is too high are considered as unreliable [19]) with the solution existence problem.

Theorem 2. *The linear equation system $A\mu = b$ introduced in the HRE approach has exactly one strictly positive solution for $0 < r \leq n - 2$ if*

$$\mathcal{K}(M) < 1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} \quad (10)$$

where $n = |C_U \cup C_K|$ - is the number of all the estimated concepts, $r = |C_K|$ - is the number of the known concepts.

Proof. Following (Def. 2), the value of Koczkodaj's inconsistency index $\mathcal{K}(M)$, in short \mathcal{K} , means that the maximal inconsistency for some triad m_{pq}, m_{qr} and m_{pr} is \mathcal{K} . Thus, in the case of an arbitrarily chosen triad m_{ik}, m_{kj}, m_{ij} it must hold that:

$$\mathcal{K} \geq \kappa_{i,j,k} = \min \left\{ \left| 1 - \frac{m_{ij}}{m_{ik}m_{kj}} \right|, \left| 1 - \frac{m_{ik}m_{kj}}{m_{ij}} \right| \right\} \quad (11)$$

This means that either: $m_{ij} \leq m_{ik}m_{kj}$ implies that $\mathcal{K} \geq 1 - \frac{m_{ij}}{m_{ik}m_{kj}}$, or $m_{ik}m_{kj} \leq m_{ij}$ implies that $\mathcal{K} \geq 1 - \frac{m_{ik}m_{kj}}{m_{ij}}$. Denoting

$$\alpha \stackrel{df}{=} 1 - \mathcal{K} \quad (12)$$

we obtain the result that either $m_{ij} \leq m_{ik}m_{kj}$ implies $m_{ij} \geq \alpha \cdot m_{ik}m_{kj}$, or $m_{ik}m_{kj} \leq m_{ij}$ implies that $\frac{1}{\alpha} \cdot m_{ik}m_{kj} \geq m_{ij}$. It is easy to see that $0 \leq \mathcal{K} < 1$, thus $0 < \alpha \leq 1$. Thus, both these assertions lead to the common conclusion:

$$\alpha \cdot m_{ik}m_{kj} \leq m_{ij} \leq \frac{1}{\alpha} m_{ik}m_{kj} \quad (13)$$

for every $i, j, k \in \{1, \dots, n\}$. This mutual relationship between entries of M can be written as the parametric equation $m_{ij} = t \cdot m_{ik}m_{kj}$ where $\alpha \leq t \leq \frac{1}{\alpha}$. Using this equation the matrix A (see 8) can be written as:

$$A = \begin{bmatrix} t_{1,1} m_{1,k} m_{k,1} & \cdots & -\frac{m_{1,k}}{n-1} \\ \vdots & \ddots & \vdots \\ -\frac{t_{k-1,1} m_{k-1,k} m_{k,1}}{n-1} & \ddots & -\frac{m_{k-1,k}}{n-1} \\ -\frac{t_{k,1} m_{k,1}}{n-1} & \cdots & 1 \end{bmatrix} \quad (14)$$

where $\alpha \leq t_{ij} \leq \frac{1}{\alpha}$, for $i, j \in \{1, \dots, k-1\}$. Hence, finally the matrix A can be written as the matrix product $A = BC$ where:

$$B = \begin{bmatrix} t_{1,1} m_{1,k} & \cdots & \cdots & -\frac{m_{1,k}}{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ -\frac{t_{k-1,1} m_{k-1,k}}{n-1} & \vdots & t_{k-1,k-1} m_{k-1,k} & -\frac{m_{k-1,k}}{n-1} \\ -\frac{t_{k,1}}{n-1} & \cdots & -\frac{t_{k,k-1}}{n-1} & 1 \end{bmatrix} \quad (15)$$

and

$$C = \begin{bmatrix} m_{k,1} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & m_{k,k-1} & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \quad (16)$$

Since both t_{ij} and m_{ij} are strictly positive, it holds that $B \in \mathcal{M}_{\mathbb{Z}}(n)$. Therefore, due to the third condition of the Theorem 1 where $D \stackrel{df}{=} I$, B is a nonsingular M -matrix if sums of all its rows are positive. In other words B is an M -matrix if each of the following inequalities (17) are true.

$$\begin{aligned} m_{1,k}(n-1)t_{1,1} - m_{1,k}(t_{1,2} + t_{1,3} + \dots + t_{1,k-1} + 1) &\geq 0 \\ m_{2,k}(n-1)t_{2,2} - m_{2,k}(t_{2,1} + t_{2,3} + \dots + t_{2,k-1} + 1) &\geq 0 \\ \dots\dots\dots &\dots\dots\dots \\ (n-1) - (t_{k,1} + t_{k,2} + \dots + t_{k,k-1}) &\geq 0 \end{aligned} \quad (17)$$

Due to the constraints introduced by the inconsistency $\mathcal{K}(M)$ the minimal and the maximal value of every t_{ij} is α and $\frac{1}{\alpha}$ correspondingly. Thus the inequalities (17) are true if the following two inequalities are satisfied:

$$(n-1)\alpha > \underbrace{\left(\frac{1}{\alpha} + \dots + \frac{1}{\alpha}\right)}_{n-r-2} + 1 \quad \text{and} \quad (n-1) > \underbrace{\left(\frac{1}{\alpha} + \dots + \frac{1}{\alpha}\right)}_{n-r-1} \quad (18)$$

where $r = n - k$ is the number of elements in C_K . In other words B is an M -matrix if the following two conditions are met:

$$f(\alpha) > 0, \quad \text{where} \quad f(\alpha) \stackrel{df}{=} (n-1)\alpha^2 - \alpha - (n-r-2) \quad (19)$$

and

$$g(\alpha) > 0, \quad \text{where} \quad g(\alpha) \stackrel{df}{=} (n-1)\alpha - (n-r-1) \quad (20)$$

By solving $f(\alpha) = 0$ and choosing the larger root¹ we obtain the result that:

$$\mathcal{K}(M) < 1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} \quad (21)$$

whilst the right, linear inequality $g(\alpha) > 0$ leads to

$$\mathcal{K}(M) < 1 - \frac{(n-r-1)}{(n-1)} \quad (22)$$

In order to decide which of these criteria are more restrictive and which should therefore be chosen, the following two cases need to be considered:

- (a) $r = n - 2$
- (b) $0 < r \leq n - 3$

When $r = n - 2$ it is easy to see that $f(\alpha) = \alpha g(\alpha)$. Thus both functions $f(\alpha)$ and $g(\alpha)$ take the 0 value for the same values of argument α . Hence, both criteria (21) and (22) are equal.

If $0 < r \leq n - 3$ it is easy to see² that the first condition (21) is more restrictive than (22), i.e. wherever (21) holds the inequality (22) is also true. In other words, to provide a guarantee that B is an M -matrix it is enough to consider the more restrictive condition (21).

The fact that B is an M -matrix implies that there exists an inverse matrix $B^{-1} \geq 0$ (Theorem 1). Hence, due to the form of the matrix C it is easy to see that the inverse matrix C^{-1} exists, thus A^{-1} exists and $A^{-1} = C^{-1}B^{-1} \geq 0$. Thus, due to the first condition of the Theorem 1, A is an M -matrix, which means that the equation $A\mu = b$ has a unique strictly positive solution. This conclusion completes the proof of the theorem.

Of course, the theorem proven above does not address the case $r = n - 1$. This is because $r = n - 1$ implies A is a scalar, hence solving $A\mu = b$ is trivial. When M is fully consistent, i.e. $\mathcal{K}(M) = 0$ and $\alpha = 1$, it is easy to see that both conditions (18) are satisfied. Thus, in such a case A is an M -matrix, and what follows $A\mu = b$ always has strictly positive solution. Several upper bounds for $\mathcal{K}(M)$ related to parameters n and r arising from the above theorem are gathered in the Table 1.

¹ The smaller root $\frac{1 - \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)} \leq 0$ for any $n = 3, 4, \dots$ and $0 < r \leq n - 2$, so it does not need to be taken into account.

² To demonstrate this please consider the sequence of inequalities $\left(\frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)}\right)^2 \geq \dots \geq \frac{4(n-1)(n-r-2)}{4(n-1)^2} \geq \left(\frac{n-r-1}{n-1}\right)^2$.

$0 \leq \mathcal{K}(M) <$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$n=3$	0.5	-	-	-	-
$n=4$	0.232	0.666	-	-	-
$n=5$	0.156	0.359	0.75	-	-
$n=6$	0.118	0.259	0.441	0.8	-
$n=7$	0.095	0.204	0.333	0.5	0.833

Table 1. The upper bounds for $\mathcal{K}(M)$ for which there is a guarantee that A is an M -matrix

Remark 1. Let us note that for any combination of r and n where $0 < r \leq n-2$, where $r, n \in \mathbb{N}_+$ the right side of (21) is greater than 0. In other words for a sufficiently low inconsistency the equation $A\mu = b$ always has a feasible solution.

To prove this (see 21) it is enough to show that for $n=3, 4, \dots$ holds:

$$\frac{\left(1 + \sqrt{1 + 4(n-1)(n-r-2)}\right)}{2(n-1)} < 1 \quad (23)$$

Since $\sqrt{1 + 4(n-1)(n-r-2)} \leq \sqrt{1 + 4(n-1)(n-3)}$, thus in particular

$$\frac{\left(1 + \sqrt{1 + 4(n-1)(n-3)}\right)}{2(n-1)} < 1 \quad (24)$$

Thus,

$$\sqrt{1 + 4(n-1)(n-3)} < 2n-3 \quad (25)$$

and

$$4(n-1)(n-3) < (2n-3)^2 - 1 \quad (26)$$

which is equivalent to

$$4(n-1)(n-3) < 4(n-1)(n-2) \quad (27)$$

Thus, for every $n > 1$ the above equation reduces to:

$$n-3 < n-2 \quad (28)$$

The last inequality is always satisfied, which proves that (23) is true for $n \geq 3$.

Remark 2. Another interesting observation is that the proof of the Theorem 2 takes into account only that entries of the matrix M , that make up the matrix A . Hence, there is no need to analyse the inconsistency for the whole matrix M . Instead, it is enough to analyse \widetilde{M} - the minor of M whose rows and columns correspond to the elements from the set of unknown concepts C_U . It also holds that $\mathcal{K}(\widetilde{M}) \leq \mathcal{K}(M)$. Thus, it may happen that the inconsistency of \widetilde{M} meets the condition (21), whilst the inconsistency of M is too high.

Assuming that $C_U = \{c_1, \dots, c_k\}$ let us define the matrix \widetilde{M} as follows:

$$\widetilde{M} = \begin{bmatrix} 1 & \cdots & m_{1,k} \\ \vdots & \ddots & \vdots \\ m_{k-1,1} & \ddots & m_{k-1,k} \\ m_{k,1} & \cdots & 1 \end{bmatrix} \quad (29)$$

It might be noticed that assuming $\alpha \stackrel{df}{=} 1 - \mathcal{K}(\widetilde{M})$ in (12) the proof of the Theorem 2 does not change. Hence, instead of exploring the inconsistency of M it is sufficient to examine the inconsistency of its minor \widetilde{M} . Thereby, the upper bounds given in the Table 1 can be applied to $\mathcal{K}(\widetilde{M})$ instead of $\mathcal{K}(M)$.

By definition of *Koczkodaj inconsistency index*, $\mathcal{K}(M)$ is the maximum of $T_M = \{\kappa_{i,j,r} : i, j, r \in \{1, \dots, n\}\}$. Similarly, $\mathcal{K}(\widetilde{M})$ is the maximum of $T_{\widetilde{M}} = \{\kappa_{i,j,r} : i, j, r \in \{1, \dots, k\}\}$, where k is the number of elements in C_U . It is easy to see that $T_{\widetilde{M}} \subseteq T_M$. This implies that also $\max T_{\widetilde{M}} \leq \max T_M$, which leads to the conclusion that $\mathcal{K}(\widetilde{M}) \leq \mathcal{K}(M)$.

4 Summary

The reliability of the results achieved in the *PC* models are inseparably linked to the degree of inconsistency of the input data [19]. The lower the inconsistency the better and more reliable the results might be expected to be. Therefore, most practical applications of the *PC* method seek to construct the *PC* matrix with the smallest possible inconsistency. The theorem proven in this article is in line with the tendency to seek *PC* solutions with low inconsistency. It shows that for an appropriately small inconsistency $\mathcal{K}(M)$ the recently proposed *Heuristic Rating Estimation* method always has a feasible solution.

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